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Nonprobabilistic, Convex-Theoretic Modeling of Scatter in Material Properties

Isaac Elishakoff,* Pierre Elisseeff,† and Stewart A. L. Glegg‡
Florida Atlantic University, Boca Raton, Florida 33431

Nonprobabilistic, convex modeling of uncertain material properties for viscoelastic structures is developed in this paper. In particular, the problem of forced vibrations of viscoelastic beams is studied. First the analytic solution by Inman is generalized for a deterministic set of variables, describing material properties. Next, these variables are treated as varying in a solid "ball" in the four-dimensional space, thus modeling the scatter in material properties. The least favorable response needed for the design of the structure is determined.

Nomenclature

$A(s)$	= Laplace transform of $a(t)$
$a(t)$	= defined in Eq. (5)
c	= viscous damping coefficient
E_0	= initial Young's modulus
$F(s)$	= Laplace transform of $f(t)$
$f(x, t)$	= excitation
$G(s)$	= Laplace transform of $g(t)$
$g(t - \tau)$	= viscoelastic kernel
I	= moment of inertia
l	= beam length
P	= radius of uncertainty ball
Q	= defined in Eq. (81)
s	= Laplace transform parameter
t	= time
$u(x, t)$	= transverse displacement
x	= axial coordinate
Z	= artificial variable
$\alpha, \beta, \gamma, \delta$	= material parameters
$\alpha_1, \beta_1, \gamma_1, \delta_1$	= defined in Eq. (55)
$\delta(t)$	= Dirac's delta function
δ_{mn}	= Kronecker's delta
ρ	= linear density
τ	= time
$\varphi_n(x)$	= normal mode
ω_n	= natural frequency

Introduction

BEHAVIOR, vibration, and stability of viscoelastic structures have been dealt with in a number of monographs.[1-3] In these studies, material properties of the structure have been fixed at some deterministic parameters. However, it is well established that the viscoelastic properties of structures exhibit a large scatter.[4] This scatter is usually accounted for by considering the material properties as random variables. Cozzarelli and Huang[5] and Huang and Cozzarelli[7] were apparently the first investigators to include material uncertainty in their analyses. Recently, Hilton et al.[8] extended the elastic-viscoelastic analogies to the stochastic case due to random linear viscoelastic material properties. Both Gaussian and beta distributions were considered for modeling the uncertainty in the data.

In probabilistic analyses, the needed probabilistic information for analysis was postulated as given. For example, Huang and Cozzarelli[7] utilize a log-normal density or a truncated log-normal density. However, extensive experimental data are needed to substantiate the probability densities with regards to the data.

More often, the necessary data are simply lacking, or only partial information is available about the parameters. In these circumstances, the usefulness of the results of probabilistic modeling when it is based on incomplete data may be questionable.

In this study we further develop the nonprobabilistic, convex modeling[9,10] for dealing with material uncertainty and attendant response variability. Note that Shinozuka[11] studied the response variability in the stochastic context. Upper bound results were derived in two cases in which the spectral density function of the stochastic field assumed limiting shapes. The importance of such response variability studies is immediately understood if one recognizes that it is rather difficult to estimate experimentally the autocorrelation function or, equivalently, the spectral density function of the stochastic variation of material properties. In this study, we consider an example of a viscoelastic beam, dealt with deterministically, for fixed parameters, by Inman.[12] The response uncertainty is directly related to the material uncertainty.

Basic Equations for Vibrating Viscoelastic Beam

Transverse vibrations of a viscoelastic beam are governed by the following differential equation:

$$\left[E_0 I \frac{\partial^4 u(x, t)}{\partial x^4} + I \int_0^t g(t - \tau) \frac{\partial^4 u(x, \tau)}{\partial x^4} \mathrm{d}\tau\right] + c \frac{\partial u(x, t)}{\partial t} + \rho \frac{\partial^2 u(x, t)}{\partial t^2} = f(x, t) \tag{1}$$

Equation (1) is supplemented by appropriate boundary and initial conditions. We shall seek the solution of the problem through separation of variables. Normal modes φ_n of the beam with constant initial modulus E_0, uniform cross section, and uniform density satisfy the equation

$$E_0 I \frac{\partial^4 \varphi_n}{\partial x^4} + \rho \frac{\partial^2 \varphi_n}{\partial t^2} = 0 \tag{2}$$

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*Professor, Department of Mechanical Engineering, and Associate Director, Center for Applied Stochastics Research.

†Graduate Research Assistant, Department of Ocean Engineering; on leave from Ecole Centrale Paris, 92290 Châtenay-Malabry, France.

‡Professor, Department of Ocean Engineering. Member AIAA.

where $u = \varphi_n(x)\exp(i\omega_n t)$, $\omega_n \equiv \beta_n^2$ so that the equation for the φ_n reads

$$E_0 I \frac{d^4\varphi_n}{dx^4} = \rho\beta_n^4\varphi_n \tag{3}$$

The normal modes satisfy the orthogonality condition

$$\int_0^l \varphi_n(x)\varphi_m(x)\,dx = \delta_{mn}$$
$$E_0 I \int_0^l \frac{d^4\varphi_n}{dx^4}\varphi_m(x)\,dx = \rho\beta_n^4\delta_{mn} \tag{4}$$

The solution of Eq. (1) is represented in the following form:

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x) \tag{5}$$

We substitute Eq. (5) into Eq. (1), multiply by $\varphi_m(x)$, and integrate over the interval $(0, l)$ to yield

$$\sum_{n=1}^{\infty} \rho\frac{d^2a_n}{dt^2}(\varphi_n, \varphi_m) + c\frac{da_n}{dt}(\varphi_n, \varphi_m) + E_0 I a_n(t)(\varphi_n^{IV}, \varphi_m)$$
$$+ I\int_0^l g(t-\tau)a_n(\tau)(\varphi_n^{IV}, \varphi_m)\,d\tau = (f, \varphi_m) \tag{6}$$

where (φ, ψ) is the inner product defined by

$$(\varphi, \psi) = \int_0^l \varphi(x)\psi(x)\,dx \tag{7}$$

Because of the orthogonality property (4), we are left with

$$\rho\frac{d^2a_m}{dt^2} + c\frac{da_m}{dt} + E_0 I\frac{(\varphi_m^{IV}, \varphi_m)}{(\varphi_m, \varphi_m)}a_m$$
$$+ I\int_0^l g(t-\tau)a_m(\tau)\frac{(\varphi_m^{IV}, \varphi_m)}{(\varphi_m, \varphi_m)}\,d\tau$$
$$= \frac{(f, \varphi_m)}{(\varphi_m, \varphi_m)} \qquad (m = 1, 2, \ldots) \tag{8}$$

or, more specifically,

$$\frac{d^2a_m}{dt^2} + \frac{c}{\rho}\frac{da_m}{dt} + \beta_m^4 a_m(t) + \beta_m^4\int_0^l \frac{1}{E_0}g(t-\tau)a_m(\tau)\,d\tau$$
$$= \frac{1}{\rho}f_m(t) \qquad [g(t) = 0, \quad t < 0] \tag{9}$$

where

$$f_m(t) = (f, \varphi_m)/(\varphi_m, \varphi_m) \tag{10}$$

The form of the differential equation, Eq. (9), is analogous but not coincident with Eq. (8) in Ref. 10.

We assume zero initial conditions. Taking the Laplace transform of Eq. (9) yields

$$\left(s^2 + \frac{c}{\rho}s + \beta_m^4\right)A_m(s) + \frac{\beta_m^4}{E_0}G(s)A_m(s) = \frac{1}{\rho}F_m(s) \tag{11}$$

We now follow Golla and Hughes[13] and Inman[12] and take the following analytical form for $G(s)$:

$$G(s) = (\alpha s^2 + \gamma s)(s^2 + \beta s + \delta)^{-1} \tag{12}$$

Some restrictions will be imposed later on, on the values of these parameters, α, β, γ, and δ. Now, $G(s)$ can be represented as

$$G(s) = \alpha\left[1 + \left(\frac{\gamma}{\alpha} - \beta\right)\frac{s}{s^2 + \beta s + \delta} - \frac{\delta}{s^2 + \beta s + \delta}\right] \tag{13}$$

so that the positive kernel function $g(t)$

$$g(t) = \alpha\left[\delta(t) + \left(\frac{\gamma}{\alpha} - \beta\right)\frac{ae^{at} - be^{bt}}{a - b}\right.$$
$$\left. - \frac{\delta}{\sqrt{D}}e^{-(1/2)\beta t}\sinh\sqrt{D}t\right] \tag{14}$$

where

$$a = \beta/2 + \sqrt{D}, \qquad b = -\beta/2 - \sqrt{D}, \qquad D = \beta^2/4 - \delta \tag{15}$$

In addition, we assume that the discriminant D is nonnegative. Substitution of Eq. (12) into Eq. (11) results in

$$\left(s^2 + \frac{c}{\rho}s + \beta_m^4 + \frac{\beta_m^4}{E_0}\frac{\alpha s^2 + \gamma s}{s^2 + \beta s + \delta}\right)a_m(s) = \frac{1}{\rho}f_m(s) \tag{16}$$

Equation (16) corresponds to an equation with the fourth degree of s; hence it can be written as two second-degree equations of the form

$$\left\{\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}s^2 + \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}s + \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}\right\}\begin{bmatrix} A_m(s) \\ Z_m(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho}F_m(s) \\ 0 \end{bmatrix} \tag{17}$$

where $Z_m(s)$ is an artificial variable (for the general issue of artificial variables one may consult studies by Ahrens,[14] McTavish,[15] McTavish et al.[16] and Ottl[17]). The matrices in Eq. (17) are taken to be symmetric to simplify the analysis. The artificial variable $Z_m(s)$ is chosen in such a way that the transfer functions in Eqs. (16) and (17) are coincident. Equation (16) reads

$$\left[s^4 + \left(\frac{c}{\rho} + \beta\right)s^3 + \left(\beta_m^4 + \frac{c\beta}{\rho} + \delta + \alpha\frac{\beta_m^4}{E_0}\right)s^2\right.$$
$$\left. + \left(\frac{c\delta}{\rho} + \beta\beta_m^4 + \frac{\beta_m^4\gamma}{E_0}\right)s + \delta\beta_m^4\right]a_m(s)$$
$$= \frac{1}{\rho}f_m(s)(s^2 + \beta s + \delta) \tag{18}$$

Equation (17) becomes

$$(m_1s^2 + c_1s + k_1)A_m + (m_2s^2 + c_2s + k_2)Z_m = F_m(s)/\rho \tag{19}$$

$$(m_2s^2 + c_2s + k_2)A_m + (m_3s^2 + c_3s + k_3)Z_m = 0 \tag{20}$$

To eliminate $Z_m(s)$ from Eqs. (19) and (20) we first note that from Eq. (20)

$$Z_m(s) = -\frac{m_2s^2 + c_2s + k_2}{m_3s^2 + c_3s + k_3}A_m(s) \tag{21}$$

We substitute Eq. (21) into Eq. (19) to yield

$$\begin{aligned}[(m_1m_3 - m_2^2)s^4 &+ (m_3c_1 + m_1c_3 - 2m_2c_2)s^3 + (m_1k_3 \\ &+ m_3k_1 + c_1c_3 - c_2^2 - 2k_2m_2)s^2 + (c_1k_3 + c_3k_1 \\ &- 2k_2c_2)s + (k_1k_3 - k_2^2)]A_m(s) \\ &= (m_3s^2 + c_3s + k_3)F_m(s)/\rho \end{aligned} \tag{22}$$

Comparison of Eqs. (18) and (22) shows that it is sufficient to choose parameters in the following form:

$$m_1 = m_3 = 1, \qquad m_2 = 0, \qquad c_3 = \beta, \qquad k_3 = \delta \tag{23}$$

In addition

$$c_1 = c/\rho, \qquad k_1 = c_2^2 + \beta_m^4 + \alpha\beta_m^4/E_0 \tag{24}$$

We are left with two additional equations:

$$\begin{aligned}\beta c_2^2 - 2k_2c_2 &= (\gamma - \alpha\beta)E_0\beta_m^4/E_0 \\ \delta c_2^2 - k_2^2 &= -\alpha\delta\beta_m^4/E_0\end{aligned} \tag{25}$$

We also assume that $\alpha\beta = \gamma$, yielding

$$k_2 = \beta c_2/2 \tag{26}$$

Equation (25) takes the form

$$(\beta^2/4 - \delta)c_2^2 = \alpha\delta\beta_m^4/E_0 \tag{27}$$

which, for definiteness, suggests that, for c_2 to be real, we should require that $\beta^2 > 4\delta$. This, however, is implied by Eq. (15). Hence we put, as is done in Ref. 10,

$$\beta^2 = 8\delta \tag{28}$$

which is in the allowable range of variation. Equation (27) then leads to

$$c_2 = \sqrt{(\alpha/E_0)}\beta_m^2 \tag{29}$$

Bearing in mind Eq. (26), we obtain

$$\begin{aligned}k_2 = \sqrt{(2\alpha\delta/E_0)}\beta_m^2, \qquad k_1 &= (1 + 2\alpha/E_0)\beta_m^4 \\ c_3 = \beta &= 2\sqrt{2\delta}\end{aligned} \tag{30}$$

The function $G(s)$ in Eq. (12) becomes

$$G(s) = \alpha(s^2 + 2s\sqrt{2\delta})(s^2 + 2s\sqrt{2\delta} + \delta)^{-1} \tag{31}$$

Effectively we are left with two parameters, α and δ. Substitution into Eq. (17) results in

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}\ddot{a}_m\\ \ddot{z}_m\end{pmatrix} + \begin{bmatrix}\dfrac{c}{\rho} & \sqrt{\dfrac{\alpha}{E_0}}\,\beta_m^2 \\ \sqrt{\dfrac{\alpha}{E_0}}\,\beta_m^2 & 2\sqrt{2\delta}\end{bmatrix}\begin{pmatrix}\dot{a}_m\\ \dot{z}_m\end{pmatrix}$$
$$+ \begin{bmatrix}\left(1 + 2\dfrac{\alpha}{E_0}\right)\beta_m^4 & \sqrt{\dfrac{2\alpha\delta}{E_0}}\,\beta_m^2 \\ \sqrt{\dfrac{2\alpha\delta}{E_0}}\,\beta_m^2 & \delta\end{bmatrix}\begin{pmatrix}a_m\\ z_m\end{pmatrix} = \begin{pmatrix}\dfrac{1}{\rho}f_m(t)\\ 0\end{pmatrix} \tag{32}$$

For specific excitation this equation should be solved.

Applications to a Simply Supported Beam

Consider now the beam that is simply supported at its ends. The mode shapes and the eigenvalues are

$$\varphi_m(x) = \sqrt{\frac{2}{l}}\sin\frac{m\pi x}{l}, \qquad \beta_m^4 = \left(\frac{m\pi}{l}\right)^4\frac{EI}{\rho} \tag{33}$$

Consider first an excitation that is spacewise uniform and timewise harmonic:

$$f(x, t) = q_0 \exp(i\omega t) \tag{34}$$

We will confine ourselves to the steady-state solution. Via Eq. (10) we obtain

$$f_m(t) = R_m \exp(i\omega t) \tag{35}$$

where

$$R_m = \sqrt{2l}q_0[1 - (-1)^m]/\mathrm{m}\pi \tag{36}$$

A steady-state solution of Eq. (32) is sought in the form

$$a_m(t) = \bar{a}_m \exp(i\omega t), \qquad z_m(t) = \bar{z}_m \exp(i\omega t) \tag{37}$$

Substituting Eqs. (37) into Eq. (32) yields

$$\bar{a}_m = \left[-\omega^2 + \frac{c}{\rho}i\omega + \left(1 + 2\frac{\alpha}{E_0}\right)\beta_m^4 + \left(\sqrt{\frac{\alpha}{E_0}}\beta_m^2 i\omega + \sqrt{\frac{2\alpha\delta}{E_0}}\beta_m^2\right)p_m\right]^{-1}\frac{1}{\rho}R_m, \qquad \bar{z}_m = P_m\bar{a}_m \tag{38}$$

$$P_m = (\omega^2 - 2\sqrt{2\delta}i\omega - \delta)^{-1}\left(\sqrt{\frac{\alpha}{E_0}}\beta_m^2\, i\omega + \sqrt{\frac{2\alpha\delta}{E_0}}\,\beta_m^2\right)$$

The steady-state response in the middle cross section is

$$U(l/2, t) = \mathrm{Re}\left\{\sum_{m=1}^{\infty}\bar{a}_m[1 - (-1)^m]e^{i\omega t}/2\right\} \tag{39}$$

where Re denotes the real part.

Consider now a case of a general nonharmonic excitation. We seek the complementary solution of Eq. (32) in the following form:

$$a_m(t) = E_1e^{rt}, \qquad z_m(t) = E_2e^{rt} \tag{40}$$

Substituting into Eq. (32) and requiring that E_1 and E_2 do not vanish simultaneously, we arrive at the following equation:

$$\begin{aligned} r^4 &+ \left(2\sqrt{2\delta} + \frac{c}{\rho}\right)r^3 + \left[\delta + \left(1 + \frac{\alpha}{E_0}\right)\beta_m^4 \right. \\ &\left. + 2\sqrt{2\delta}\frac{c}{\rho}\right]r^2 + \left[\frac{c}{\rho}\delta + 2\sqrt{2\delta}\left(1 + \frac{\alpha}{E_0}\right)\beta_m^4\right]r \\ &+ \beta_m^4\delta = 0 \end{aligned} \tag{41}$$

We first study the behavior of the roots. It can be shown, through use of the Routh-Hurwitz test, that the equation

$$r^4 + \lambda_3r^3 + \lambda_2r^2 + \lambda_1r + \lambda_0 = 0 \tag{42}$$

has roots with negative real parts, if and only if

$$\lambda_1\lambda_2\lambda_3 > \lambda_1^2 + \lambda_0\lambda_3^2 \tag{43}$$

It can be verified that this condition is satisfied. It should be noted that inequality (43) is also satisfied for the vanishing viscous damping. This implies that the viscoelastic behavior is introducing an additional damping to the structure. Since the roots of the characteristic equation have negative real parts, the complementary solution vanishes when $t \to \infty$. We are interested in the behavior of the displacement after sufficient time has elapsed from the start. In this case the complimentary solution can be discarded. For the excitation

$$f(t) = bt \tag{44}$$

Eq. (10) yields

$$f_m(t) = h_m t, \qquad h_m = \sqrt{2l}b[1 - (-1)^m]/\rho m\pi \tag{45}$$

The particular solution of Eq. (32) is given in the form

$$a_m(t) = S_1 t + S_2, \qquad z_m(t) = G_1 t + G_2 \tag{46}$$

Substitution into Eq. (32) results in

$$S_1 = \frac{h_m}{\beta_m^4}, \qquad S_2 = -\left(f2\sqrt{\frac{2}{\delta}}\frac{\alpha'}{E_0\beta_m^4} + \frac{c}{\rho\beta_m^8}\right)h_m$$

$$G_1 = -\sqrt{\frac{2\alpha}{\delta E_0}}\frac{h_m}{\beta_m^2} \tag{47}$$

$$G_2 = \frac{h_m}{\beta_m^2}\sqrt{\frac{\alpha}{E_0\delta}}\left(\frac{3}{\sqrt{\delta}} + \frac{4}{\sqrt{\delta}}\frac{\alpha}{E_0} + \frac{\sqrt{2}c}{\rho\beta_m^4}\right)$$

The solution for $a_m(t)$ can be written as

$$a_m(t) = \frac{h_m}{\beta_m^4}\left(t - \frac{1}{\rho\beta_m^4}c_{\text{eq}}\right) \tag{48}$$

where c_{eq} is an equivalent viscous damping, representing the combined effect of both genuine viscous damping and additional damping introduced by the viscoelastic behavior:

$$c_{\text{eq}} = c + \tilde{c}, \qquad \tilde{c} = 2\rho\sqrt{2/\delta}\alpha\beta_m^4/E_0 \tag{49}$$

Note that, as expected, for $\alpha = \delta = \chi$ and χ tending to zero, the equivalent viscous damping c_{eq} reduces to the genuine viscous damping c, since the contribution due to viscoelasticity vanishes. The nondimensional response reads

$$\underline{\tilde{a}_m}(t) = a_m(t)/\tilde{r} = K_1(K_2T - 1) \tag{50}$$

where $\tilde{r} = \sqrt{I/A}$ is radius of inertia, $T = \omega_m t$ is nondimensional time, and

$$K_1 = \frac{h_m c_{\text{eq}}}{\rho\beta_m^8\tilde{r}} \qquad K_2 = \frac{\rho\beta_m^2}{c_{\text{eq}}} \tag{51}$$

Finally,

$$U(\ell/2, t) = \sum_{m=1}^{\infty}\tilde{a}_m(t)\sin(m\pi) \tag{52}$$

Least and Most Favorable Responses

The general methodology for convex modeling of uncertainty was developed in Refs. 9 and 10. Hereinafter, we will apply this methodology for modeling uncertainty in material properties of a viscoelastic structure. We discuss the realistic situation, assuming that we possess only some limited information on material parameters α, β, γ, and δ, namely, that they belong to some convex set. We denote the displacement in either Eq. (39) or Eq. (52) as follows:

$$u(\ell/2, t) = f(\alpha, \beta, \gamma, \delta) \tag{53}$$

We expand function $f(\alpha, \beta, \gamma, \delta)$ in Taylor series and take into account only the linear terms

$$f(\alpha, \beta, \gamma, \delta) \simeq f(\alpha_0, \beta_0, \gamma_0, \delta_0) + \varphi^T\zeta$$
$$\varphi = \nabla f|_{\alpha=\alpha_0,\beta=\beta_0,\gamma=\gamma_0,\delta=\delta_0} \tag{54}$$

and

$$\zeta = \left(\frac{\alpha - \alpha_0}{\alpha_1}, \frac{\beta - \beta_0}{\beta_1}, \frac{\gamma - \gamma_0}{\gamma_1}, \frac{\delta - \delta_0}{\delta_1}\right) \equiv (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \tag{55}$$

where α_0, β_0, γ_0, and δ_0 are some average parameters. In addition, parameters α_1, β_1, γ_1, and δ_1 are introduced for dimensionless formulation; in particular, the following values have been taken: $\alpha_1 = 1$ N/m^2, $\beta_1 = 1$ s^{-1}, $\gamma_1 = 1$ N/m^{2}s, $\delta_1 = 1$ s^{-2}. The scatter is modeled as the vector ζ belonging to some set $Z(P)$ taken as a four-dimensional solid ball given by

$$Z(P) = (\zeta|\zeta^T\zeta \le P^2) \tag{56}$$

where P is the radius of the sphere. It should be noted that the derivation of radius P assumes that the parameters α, β, γ, and δ are of similar order of magnitude and that the variations of these values are also of similar magnitude. On the other hand, McTavish[15] gives a numerical example (taken from Bagley and Torvik[19]) that demonstrates that these parameters have multiple orders of magnitude difference. However, this may be simply remedied by choosing α_1, β_1, γ_1, and δ_1 such that an approximate distribution may be represented by a sphere in the appropriately transformed coordinates. We are interested in the maximum possible value the function $f(\alpha, \beta, \gamma, \delta)$ may take in this solid ball. This will result in the least favorable response the system may experience when the parameters vary in the solid ball, Eq. (56). We will also be interested in the minimum value of $f(\alpha, \beta, \gamma, \delta)$. This will be identified with the best possible response.

We follow Ref. 10 to solve this problem. Although the physical context of Ref. 10 is different (it dealt with the uncertain initial imperfections of shells), the mathematical approach is similar. We are interested in evaluating the extrema of the function $f(\alpha, \beta, \gamma, \delta)$ within the solid ball, given in Eq. (56):

$$\mu_j(P) = \operatorname*{ext}_{\zeta\in Z} f(\alpha, \beta, \gamma, \delta) \tag{57}$$

where $j = 1$ for maximum and $j = 2$ for minimum. Since $Z(P)$ is convex, $f(\alpha, \beta, \gamma, \delta)$ reaches its maximum and minimum values on the boundary, i.e., on the set

$$B(P) = \{\eta\colon \eta^T\eta = P^2\} \tag{58}$$

We define the Lagrangian as follows, in view of Eq. (58):

$$L = f(\alpha_0, \beta_0, \gamma_0, \delta_0) + \varphi^T\eta + \lambda(\eta^T\eta - P^2) \tag{59}$$

We demand that

$$\partial L/\partial\eta = \varphi + 2\lambda\eta = 0 \tag{60}$$

yielding

$$\eta = -\varphi/2\lambda \tag{61}$$

However, in view of the equality $\eta^T\eta = P^2$, we arrive at

$$\lambda = \pm\sqrt{\varphi^T\varphi}/2P, \qquad \eta = \pm P\varphi/\sqrt{\varphi^T\varphi} \tag{62}$$

where the plus sign is associated with the maximum and the minus sign with the minimum. We further substitute the expression (62) for η instead of ζ in Eq. (54).

Therefore the final results for the least favorable response (denoted as LFR) and for the most favorable response (indicated as MFR) are

$$\text{LFR} = \mu_1(P) = f_0 + P\sqrt{\varphi^T\varphi} \tag{63}$$

$$\text{MFR} = \mu_2(P) = f_0 - P\sqrt{\varphi^T\varphi} \tag{64}$$

where

$$\varphi^T = \left(\frac{\partial f}{\partial\alpha}, \frac{\partial f}{\partial\beta}, \frac{\partial f}{\partial\gamma}, \frac{\partial f}{\partial\delta}\right)_0, \qquad f_0 = f(\alpha_0, \beta_0, \gamma_0, \delta_0) \tag{65}$$

Equation (53) can be put in the following form, in conjunction with Eq. (35),

$$f(\alpha, \beta, \gamma, \delta) = (\Psi e^{i\omega t} + \Psi^* e^{-i\omega t})/2 \tag{66}$$

where asterisk denotes complex conjugate. The function Ψ, with term $m = 1$, retained is

$$\bar{a}_1 = B_1/C_1, \qquad B_1 = 2\sqrt{2l}q_0(\delta - \omega^2 + i\beta\omega)/\pi\rho \tag{67}$$

The derivatives are calculated at $\alpha = \alpha_0$, $\beta = \beta_0$, $\gamma = \gamma_0$, and $\delta = \delta_0$, and the asterisk denotes complex conjugate. Now

$$\frac{\partial\Psi}{\partial\alpha} = \frac{1}{C_1^2}\left[\frac{4q_0}{\pi\rho}(\delta - \omega^2 + i\beta\omega)\right]\left[\frac{I}{\rho}\left(\frac{\pi}{\ell}\right)^4\omega^2\right]$$

$$\frac{\partial\Psi}{\partial\beta} = \frac{4q_0 i\omega}{\pi C_1^2\rho}\left\{C_1 - (\delta - \omega^2 + i\beta\omega)\right.$$
$$\left.\times\left[-\omega^2 + \frac{E_0 I}{\rho}\left(\frac{\pi}{\ell}\right)^4 + \frac{ic\omega}{\rho}\right]\right\} \tag{68}$$
$$\frac{\partial\Psi}{\partial\gamma} = -\frac{1}{C_1^2}\left[\frac{4q_0}{\pi\rho}(\delta - \omega^2 + i\beta\omega)\right]\frac{i\omega}{\rho}I\left(\frac{\pi}{\ell}\right)^4$$

$$\frac{\partial\Psi}{\partial\delta} = \frac{4q_0}{\pi C_1^2\rho}\left\{C_1 - (\delta - \omega^2 + i\beta\omega)\right.$$
$$\left.\times\left[-\omega^2 + \frac{E_0 I}{\rho}\left(\frac{\pi}{\ell}\right)^4 + \frac{i\omega c}{\rho}\right]\right\}$$

Equations (63) and (64) become

$$\text{LFR} = f_0 - P\left[\left|\frac{\partial\Psi_0}{\partial\alpha}\right|^2 + \left|\frac{\partial\Psi_0}{\partial\beta}\right|^2 + \left|\frac{\partial\Psi_0}{\partial\gamma}\right|^2 + \left|\frac{\partial\Psi_0}{\partial\delta}\right|^2\right]^{1/2} \tag{69}$$

$$\text{MFR} = f_0 + P\left[\left|\frac{\partial\Psi_0}{\partial\alpha}\right|^2 + \left|\frac{\partial\Psi_0}{\partial\beta}\right|^2 + \left|\frac{\partial\Psi_0}{\partial\gamma}\right|^2 + \left|\frac{\partial\Psi_0}{\partial\delta}\right|^2\right]^{1/2} \tag{70}$$

In the case of the timewise linearly increasing excitation given in Eq. (44), the least favorable response as a function of nondimensional time T reads, in view of Eq. (50),

$$\text{LFR}(T) = a_m(T)\big|_{\substack{\alpha=\alpha_0\\ \delta=\delta_0}} + P\left[\left|\frac{\partial a_m}{\partial\alpha}\right|_0^2 + \left|\frac{\partial a_m}{\partial\delta}\right|_0^2\right]^{1/2} \tag{71}$$

$$\text{MFR}(T) = a_m(T)\big|_{\substack{a=a_0\\ \delta=\delta_0}} - P\left[\left|\frac{\partial a_m}{\partial\alpha}\right|_0^2 + \left|\frac{\partial a_m}{\partial\delta}\right|_0^2\right]^{1/2} \tag{72}$$

4

For numerical values $h_m = \beta_m$ we have

$$\frac{\partial a_m}{\partial\alpha} = -2\frac{\sqrt{2}}{\delta}\frac{\omega_m}{E_0}, \qquad \frac{\partial a_m}{\partial\delta} = \frac{1}{\delta}\frac{\sqrt{2}}{\delta}\frac{\alpha}{E_0}\omega_m \tag{73}$$

Numerical Examples and Discussion

Consider first the numerical results for the case of the harmonic excitation. Figure 1a portrays the dependence of the least and most favorable responses on the uncertainty radius P. The parameters were fixed at

$$\alpha_0^{(a)} = 34.35\times10^9\ \text{N/m}^2, \qquad \delta_0^{(a)} = 7.8125\times10^5\ \text{s}^{-2}$$
$$m = 1, \qquad E_0^{(a)} = 68.9\times10^9\ \text{N/m}^2$$
$$\rho^{(a)} = 27\ \text{kg/m}, \qquad [\beta_1^{(a)}]^4 = 3.0688\times10^5 \tag{74}$$
$$I^{(a)} = 10^{-4}\ \text{m}^4, \qquad l^{(a)} = 3\ \text{m}$$

The value of the viscous damping was fixed at $c^{(a)} = 5983.2$ kg/m s, corresponding to coefficient $\zeta = c/2\rho\omega_1 = 0.2$. As is seen in Fig. 1a, the amplitude of vibration for the nominal structure is 1.424×10^{-3} m. For the beam without viscoelastic effects, the response of the structure, evaluated through the use of the expression

$$Y_2 = \frac{4q_0}{\pi|EI(\pi/l)^4 - \omega^2\rho + i\omega c|} \tag{75}$$

is 1.979×10^{-3} m$^{3/2}$. This is 28% more than the nominal response of the viscoelastic beam. The least favorable response linearly increases with P, whereas the most favorable response linearly decreases with P. At $P = 1000$, for example, LFR $= 1.293\times10^{-3}$ m$^{3/2}$, i.e., about 9% higher than the nominal response. Figure 1b is associated with the same data as in Fig. 1a except that now $\alpha_0^{(b)} = 2\alpha_0^{(a)}$. In Fig. 1c the value $\alpha_0^{(c)} = \alpha_0^{(a)}$, but the nominal value of δ is twice as much as in Fig. 1a, namely, $\delta_0^{(c)} = 2\delta_0^{(a)}$. In Fig. 1d the data are as in Eq. (68) except $\alpha_0^{(d)} = 2\alpha_0^{(a)}$ and $\delta_0^{(d)} = 2\delta_0^{(a)}$

Figure 2 depicts the behavior of the absolute value Φ of the gradient of the function $f(\alpha, \beta, \gamma, \delta)$, namely, $\Phi = \sqrt{\varphi^T\varphi^*}$ defining the least and most favorable responses in Eqs. (63) and (64). As is seen for large values of α and small values of δ, the values of Φ can be significant, contributing to larger values of the least favorable response. On the other hand, for materials with small values of α and relatively large values of δ, the effect of uncertainty will be less pronounced.

Figure 3 is associated with the response to timewise linearly increasing excitation. It should be noted that the scatter in the responses stems from the constant term in Eq. (48). Remarkably, the viscoelastic beam and its elastic counterpart share the same slope in the asymptotic response, when $t\to\infty$. In these circumstances, however, use of the nonlinear governing equations is necessary, since the response becomes too large to justify employing the linearized analysis. The data in Fig. 3 are as in Fig. 1a except $\alpha_0^{(3)} = E_0$ and $\delta_0^{(3)} = 10\ \text{s}^{-2}$; in addition the uncertainty radius is fixed at $P = 2$. Curve 1 represents the LFR, whereas curve 2 represents the MFR. The shaded area is a region of the response variability. Equations (48) in conjunction with Eqs. (65) and (67) were utilized for obtaining the MFR and the LFR. It is noteworthy that

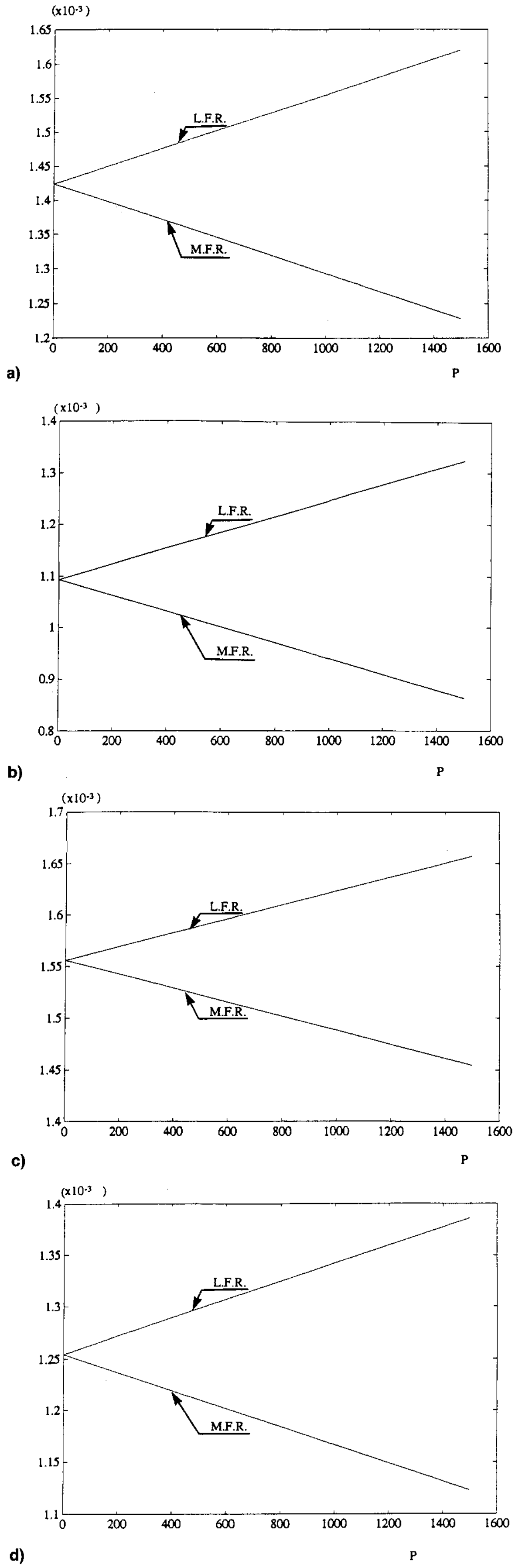

Fig. 1 **Dependence of the least favorable response (LFR) and most favorable response (MFR) upon uncertainty radius** P**: a)** $\alpha_0^{(a)} = 34.35 \times 10^9$ **N/m²,** $\delta_0^{(a)} = 7.8125 \times 10^5\ s^{-2}$**; b)** $\alpha_0^{(b)} = 2\alpha_0^{(a)}$**,** $\delta_0^{(b)} = \delta_0^{(a)}$**; c)** $\alpha_0^{(c)} = \alpha_0^{(a)}$**,** $\delta_0^{(c)} = 2\delta_0^{(a)}$**; and d)** $\alpha_0^{(d)} = 2\alpha_0^{(a)}$**,** $\delta_0^{(d)} = 2\delta_0^{(a)}$**.**

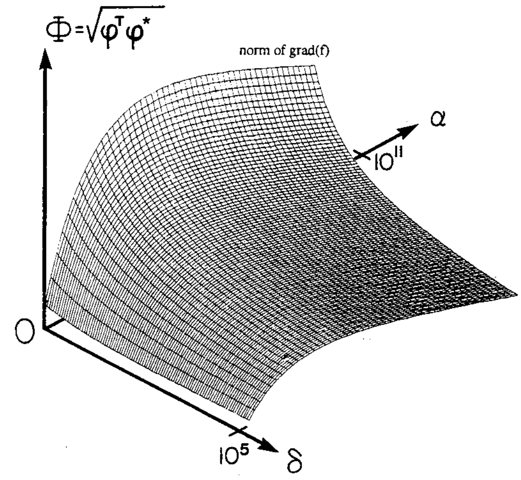

Fig. 2 **Absolute value** $\Phi = \sqrt{\varphi^T\varphi^*}$ **of the gradient of the function** $f(\alpha, \beta, \gamma, \delta)$ **as a function of parameters** α **and** δ**.**

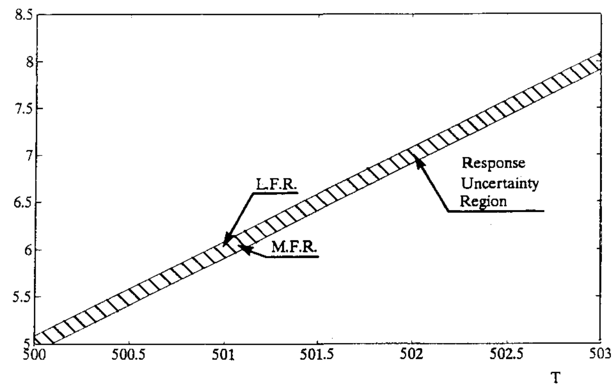

Fig. 3 **Response variability region in the beam subjected to timewise linearly increasing excitation** $[\alpha_0^{(3)} = E_0, \delta_0^{(3)} = 10\ s^{-2}]$**.**

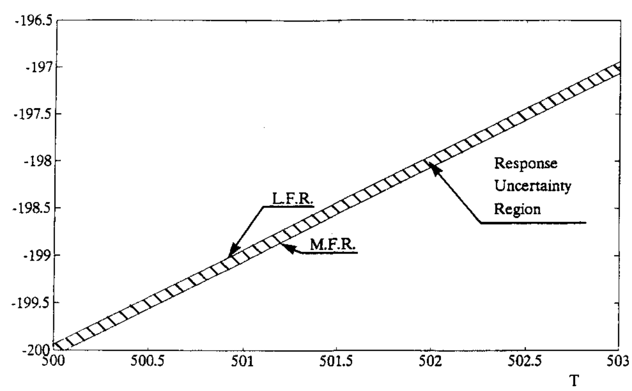

Fig. 4 **Response variability region in the beam subjected to timewise linearly increasing excitation** $[\alpha_0^{(4)} = 2\alpha_0^{(3)}, \delta_0^{(4)} = 2\delta_0^{(3)}]$**; response variability region is hatched.**

the contribution of the viscoelastic effects to the equivalent viscous damping c_{eq} in Eq. (49) through $\tilde{c}$ is much more important than that of the purely viscous damping for the numerical data chosen in Fig. 3. Therefore the "time shift" predicted by the second term in Eq. (48) is non-negligible. In Fig. 4, the data were changed to $\alpha_0^{(4)} = 2\alpha_0^{(3)}$ and $\delta_0^{(4)} = 2\delta_0^{(3)}$ with the same value of uncertainty radius. As is seen, the response variability is reduced, in comparison with Fig. 2. In Fig. 5 the data are as in Fig. 1a except that now $\alpha_0^{(5)} = \alpha_0^{(3)}/2$ and $\delta_0^{(5)} = \delta_0^{(3)}/2$.

The response variability region is now increased. This is understandable since the relative measure of uncertainty (RMU) could conveniently be defined as

$$\text{RMU} = P/Q \tag{76}$$

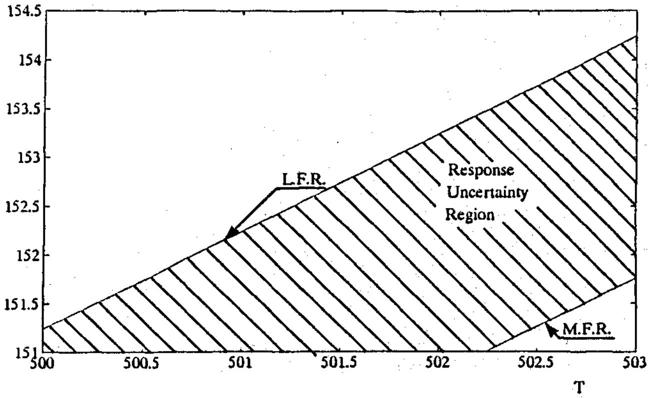

Fig. 5 Response variability region in the beam subjected to timewise linearly increasing excitation [$\alpha_0^{(5)} = \alpha_0^{(3)}/2$, $\delta_0^{(5)} = \delta_0^{(3)}/2$]; response variability region is hatched.

where P is the radius of input uncertainty, and Q is the measure of the nominal location of the uncertain vector; Q can be taken as

$$Q = \sqrt{\left(\frac{\alpha_0}{\alpha_1}\right)^2 + \left(\frac{\beta_0}{\beta_1}\right)^2 + \left(\frac{\gamma_0}{\gamma_1}\right)^2 + \left(\frac{\delta_0}{\delta_1}\right)^2} \quad (77)$$

where α_1, β_1, γ_1, and δ_1 are defined in Eq. (55).

With increasing the RMU one would anticipate the increase in the response uncertainty. This qualitative observation is in agreement with obtained results. Moreover, this study provides a rigorous way to quantify the response variability if scant information on variability is available.

Conclusions

A new, nonprobabilistic method to account for the material variability is presented in this study. Instead of describing the material parameters as random variables, as is done conventionally, we describe the material uncertainty within set-convex modeling: the material parameters are assumed to belong to a solid ball, with its center representing the nominal behavior and its radius modeling the variability. The paper presents a novel way to predict both the least favorable response and the most favorable response. The least favorable response, rather than the nominal one, should be incorporated in the engineering design of structures that possess material uncertainties.

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